Game Theory

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Simple Games

Introduction

- Five goals for this session
  1. Show iterated elimination of dominated strategies
  2. Demonstrate backwards induction
  3. Explain mixed strategies
  4. Introduce cut-point strategies
  5. Application: campaign platforms

Normal-Form Games

- In normal-form games, players make simultaneous moves
- Cells contain utilities for outcomes produced when players adopt the relevant combination of strategies
- Typically, 1 chooses rows, 2 columns (and 3 matrices)

	σ_{2a}	σ_{2b}
σ_{1a}	$(u_1(X_{aa}), u_2(X_{aa}))$	$(u_1(X_{ab}), u_2(X_{ab}))$
σ_{1b}	$(u_1(X_{ba}), u_2(X_{ba}))$	$(u_1(X_{bb}), u_2(X_{bb}))$

Iterated Elimination of Dominated Strategies

- Let σ_i denote some strategy for i and σ_j a strategy for j
- Let (σ_i, σ_j) denote a strategy pair
- Then σ_i is strictly dominated by σ_i' iff $u_i(\sigma_i, \sigma_j) < u_i(\sigma_i', \sigma_j) \; \forall \; \sigma_j$
- Normal-form games can often by solved by eliminating these from consideration (iteratively, if needed)

	L	R
U	3, 3	1, 4
M	0, 4	-1, 7
D	4, 1	2, 2

Nash Equilibria

- Strategy pair (σ_i^*, σ_j^*) represents a Nash equilibrium iff
  - $u_i(\sigma_i^*, \sigma_j^*) \geq u_i(\sigma_i', \sigma_j^*) \ \forall \ \sigma_i'$ and
  - $u_j(\sigma_i^*, \sigma_j^*) \geq u_j(\sigma_i^*, \sigma_j') \ \forall \ \sigma_j'$

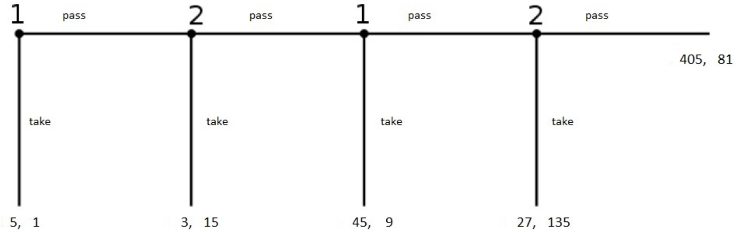
- If a single strategy pair survives IEDS, guaranteed to be a NE
- Not all NE can be identified this way
- No guarantee that a single strategy pair survives IEDS

A Model of Coordination

	sub zero	$\overline{\text{TOP BAR}}$
sub zero	$\underline{\beta}, \overline{\beta}$	0, 0
$\overline{\text{TOP BAR}}$	0, 0	$\overline{\beta}, \underline{\beta}$

Extensive-Form Games

- In extensive-form games, players make sequential moves
- Utilities appear below terminal nodes



Solving Extensive-Form Games

- If complete & perfect info, apply backwards induction
  - Begin at a terminal node and determine optimal choice
  - Do that for all terminal nodes
  - Then move up one level and repeat
  - Assume actors are forward-looking

- Example: centipede game from previous slide
  - If 2 takes, they get 135
  - If 2 passes, they get 81
  - So 2 will take
  - Anticipating this, 1 knows they get 27 if they pass
  - But they get 45 if they take
  - Anticipating that, 2 compares 15 to 9
  - And anticipating that, 1 takes 5

Subgame Perfect Equilibria

- In a sugbame perfect equilibrium, all players adopt strategies that survive backwards induction
- More formally, if the strategy profile constitutes a NE in every proper subgame of the full game, then it identifies a SPNE
- A subgame consists of a node and everything after
- There are 4 subgames to the centipede game we just solved
- SPNE are a subset of NE

Mixed Strategies

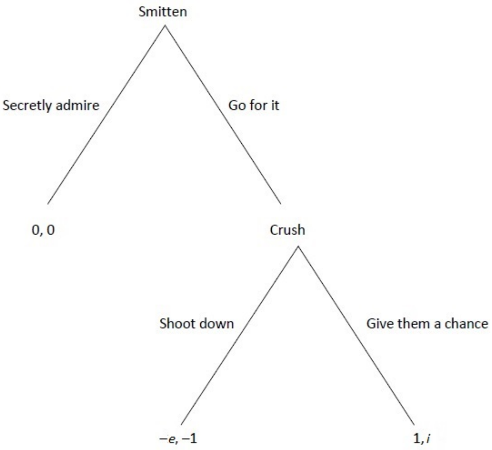
- Some games only have equilibria in mixed strategies
- When playing a mixed strategy, there are two or more pure strategies that are played with positive probability
- Players may mix when indifferent between pure strategies
- Players can induce indifference in one another by selecting particular mixing probabilities
- Consider following example, a modified Colonel Blotto

	Mountains	Plains
Mountains	-2, 1	1, -1
Plains	2, -2	-1, 2

Analysis

- Suppose A attacks D through mountains with probability q
    - $E(u_D(M)) = E(u_D(P))$
    - $\Rightarrow q(1) + (1 - q)(-2) = q(-1) + (1 - q)(2)$
    - $\Rightarrow 3q - 2 = -3q + 2$
    - $\Rightarrow 6q = 4$
    - Which simplifies to $q = \frac{2}{3}$
- Suppose D defends the mountains with probability p
    - $E(u_A(M)) = E(u_A(P))$
    - $\Rightarrow p(-2) + (1 - p)(1) = p(2) + (1 - p)(-1)$
    - $\Rightarrow -3p + 1 = 3p - 1$
    - $\Rightarrow 2 = 6p$
    - Which simplifies to $p = \frac{1}{3}$
- Giving us a MSNE of $(\frac{2}{3}M, \frac{1}{3}P; \frac{1}{3}M, \frac{2}{3}P)$

A Model of Teenage Angst



Analysis

- Begin with Crush's decision
  - u_C(a chance) $\geq$ u_C(shoot down)
  - $\Rightarrow i \geq -1$
  - Crush follows a cut-point strategy

- Then ask what Smitten does when $i \geq -1$
  - u_S(go for it$|i \geq -1$) $\geq$ u_S(secretly admire)
  - $\Rightarrow 1 \geq 0$

- And when $i < -1$
  - u_S(go for it$|i < -1$) $\geq$ u_S(secretly admire)
  - $\Rightarrow -e \geq 0$

Application: Campaign Platforms

- Candidates L and R compete in an election
- Simultaneously set $x_i \in [-1, 1]$
- Elections determined by coin toss in event of tie
- Winner receives payoff of 1, loser 0
- Voter j backs i iff $|x_i - v_j| < |x_{\neg i} - v_j|$
- Let v_m denote median value of v_j
- Unique NE: L sets $x_L = v_m$; R sets $x_R = v_m$
- Proof by contradiction
  - $E(u_L(x_L = v_m | x_R = v_m)) = 0.5$, $E(u_L(x_L \neq v_m | x_R = v_m)) = 0$
  - Because game is symmetric, same holds for R

Implications

- This is an example of a foundational model
- In reality, platforms rarely converge so perfectly
- More advanced models identify factors that explain divergence
- Yet, stylized as this model is, it yields a powerful insight
- Explains why many voters feel party is too moderate
- Strong correlation b/w prefs of MV & local expenditures
- Placement of fast food chains, petrol stations

Extension: Voter Abstention

- Now assume that voter j's strategy is to:
  - vote L if $|x_L - v_j| < \alpha_j$ & $|x_L - v_j| < |x_R - v_j|$
  - vote R if $|x_R - v_j| < \alpha_j$ & $|x_R - v_j| < |x_L - v_j|$
  - abstain if $|x_i - v_j| \geq \alpha_j$ & $|x_i - v_j| < |x_{\neg i} - v_j|$
  - where α_j is decreasing in $|v_j - v_m|$
- Platforms may no longer converge on v_m
  - x_is closer to $v_m \uparrow \#$ of voters who prefer i to $\neg i$
  - x_is farther from $v_m \downarrow \#$ of voters who abstain
  - Optimal platform balances these two effects