Game Theory

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Mathematical Foundations

Introduction

- Five goals for this session
  1. Review basic algebra
  2. Give primer on formal logic
  3. Discuss notational conventions
  4. Cover essentials of calculus
  5. Application: provision of public goods

What and Why

- Elementary algebra differs from arithmetic in that it relies upon symbolic expressions rather than numbers
- Forms fundamental basis for advanced algebra (linear algebra, abstract algebra, algebraic number theory, etc)
- Variable: a symbolic expression whose value is not fixed
- Parameter: a symbolic expression whose value is constant, but not necessarily known to us as analysts
- Ex: $Y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \ldots + \epsilon$
- Statistical analysis concerned with estimation of parameters
- Game theory concerned with construction of logical arguments

Motivation

- When you first encountered algebra, your goal was probably to solve equations with one or two unknowns
- In GT models, this is almost never the case
- However, we will frequently wish to solve equations or inequalities for a single variable in order to derive cutpoints
- Cutpoint: A critical value of a variable, typically used to determine equilibrium strategies
- Ex: Player 1 plays strategy σ_1 if $\alpha \leq \hat{\alpha}$, σ_2 if $\alpha > \hat{\alpha}$

Solving Equations and Inequalities

- Properties of Addition and Multiplication
    - Commutative: $a + b = b + a$, $a \cdot b = b \cdot a$
    - Associative: $a(b \cdot c) = (a \cdot b)c = a \cdot b \cdot c$
    - Distributive: $a(b + c) = a \cdot b + a \cdot c$
    - Remember PEMDAS

- Basic properties of equations/inequalities
    - Addition: can add (or subtract) any quantity from both sides
    - Multiplication: can multiply both sides by any quantity
    - Division: can divide both sides by any non-zero quantity
    - Sign reverses when multiplying/dividing by negative quantities

Fractions

- Basics
  - ratio $\frac{a}{b}$
  - proportion $\frac{a}{a+b}$
  - percentage $\frac{a}{a+b} \times 100$
- Common errors
  - Inappropriate cancellations
  - Breaking up additions

Validity

- Validity: property possessed by arguments whose conclusions must be true *if* their premises are all simultaneously true
- Soundness: property possessed by logically valid arguments whose premises *are* all true

- Example
  - P1: My last submission was accepted by either the APSR or the AJPS
  - P2: My last submission was rejected by the AJPS
  - C: It was accepted by the APSR
  - This argument is valid, but (sadly) not sound

Basic Rules

Modus Ponens	$[p \Rightarrow q] \wedge p \Rightarrow q$
Modus Tollens	$[p \Rightarrow q] \wedge \neg q \Rightarrow \neg p$
Hypothetical Syllogism	$[p \Rightarrow q] \wedge [q \Rightarrow t] \Rightarrow [p \Rightarrow t]$
Disjunctive Syllogism	$[p \vee q] \wedge \neg q \Rightarrow p$
De Morgan's Law	$\neg[p \wedge q] \Rightarrow [\neg p \vee \neg q]$
	and $\neg[p \vee q] \Rightarrow [\neg p \wedge \neg q]$
Contraposition	$[p \Rightarrow q] \Rightarrow [\neg q \Rightarrow \neg p]$

Rules for Quantificational Logic

Universal Instantiation	If $(\forall x \in X)P(x)$, we can infer $P(y)$ for any $y \in X$
Existential Instantiation	If $(\exists x \in X)P(x)$, we can stipulate $(y \in X)P(x)$ for some y
Negation Laws	$(\neg\exists x \in X)P(x) \Rightarrow (\forall x \in X)\neg P(x)$ and $(\neg\forall x \in X)P(x) \Rightarrow (\exists x \in X)\neg P(x)$

Common Fallacies

Irrelevant Conclusion	appeal to authority *argumentum ad hominem* genetic fallacy
Hasty Generalization	$(\exists x \in X)P(x) \not\Rightarrow (\forall x \in X)P(x)$
Affirming the Consequent	$[p \Rightarrow q] \wedge q \not\Rightarrow p$
Denying the Antecedent	$[p \Rightarrow q] \wedge \neg p \not\Rightarrow \neg q$
post hoc ergo propter hoc	Temporal precedence $\not\Rightarrow$ causation
cum hoc ergo propter hoc	Correlation $\not\Rightarrow$ causation

Proofs

- Tables with numbered propositions very rare
- Rely on rules of formal logic w/o explicitly invoking them
- Establish existence or uniqueness of equilibria through combinations of basic rules, quantificational logic

- Common strategies
  - Exhaustion—analyze all possible cases
  - Contradiction—assume proposition does not hold, then demonstrate that resulting argument contains a contradiction
  - Construction—demonstrate possibility by providing example
  - Mathematical Induction—prove a base case, then show that *if* it holds for base case, it must hold more generally

Universal

$\square$	completed proof	$\neg$	negation symbol
$\in$	is an element of	$\ni$	such that
$\forall$	for all	$\exists$	there exists
$\emptyset$	empty set	$\{a, b\}$	set of elements
(a, b)	open interval	$[a, b]$	closed interval
$\subset$	is subset of	$\cup$	union of sets
$\cap$	intersection of sets	$\setminus$	subtract from set
$\Rightarrow$	"then"	$\Leftrightarrow$	"if and only if" (iff)
$\max\{\}$	maximum of set	$\underset{x}{\operatorname{argmax}} f(x)$	x that maximizes $f(x)$

GT Conventions

- x^* typically denotes value of x chosen in equilibrium
- Superscripts *sometimes* denote special values
- Ex: $x_s^e \subset x_s \subset x$
- In statistics, $\overline{x}$ denotes x's mean value, $\hat{x}$ an estimate
- In GT, $\overline{x}$ and $\hat{x}$ typically denote special values, often cutpoints
- Same goes for $\underline{x}$, $\tilde{x}$, $\check{x}$, etc
- Usage should be clear from context

Limits

- Suppose we wish to know what value $f(x)$ tends towards as x gets arbitrarily close to 3 for $f(x) = \frac{x^2 - 9}{x - 3}$
  - That is, we wish to evaluate $\lim_{x \to 3} f(x)$
  - For $x = 3.5, f(x) = 6.5$, for $x = 2.5, f(x) = 5.5$
  - For $x = 3.1, f(x) = 6.1$, for $x = 2.9, f(x) = 5.9$
  - It looks like $f(x)$ is moving ever closer to 6 as x approaches 3
  - But for $x = 3$, $f(x)$ is undefined, because the denominator is 0
- Sometimes, it is true that $\lim_{x \to a} f(x) = f(a)$
- But, as this example demonstrates, that doesn't always work

Precise Definition of a Limit

- Again, let $f(x) = \frac{x^2 - 9}{x - 3}$, and further let $g(x) = x + 3$
  - Then $f(x) \Leftrightarrow g(x) \; \forall \; x \neq 3$
  - This is because $\frac{(x-3)(x+3)}{x-3} = x + 3 \; \forall \; x \neq 3$
  - We can thus use direct substitution to show $\lim_{x \to 3} f(x) = 6$
- This brings us to the precise definition of a limit:
  - $\lim_{x \to a} f(x) = L$ if for every $\epsilon > 0 \; \exists \; \delta > 0 \ni 0 < |x - a| < \delta \Leftrightarrow |f(x) - L| < \epsilon$
  - Implies $\lim_{x \to a^+}$ must equal $\lim_{x \to a^-}$ for $\lim_{x \to a} f(x) = L$ to exist
  - This will not be true at discontinuities

Example

- Prove $\lim_{x \to 0^+} \sqrt{x} = 0$
- We want to find $\delta > 0 \ni 0 < x < \delta \Leftrightarrow |\sqrt{x} - 0| < \epsilon$
- First, simplify: $|\sqrt{x} - 0| < \epsilon \Leftrightarrow \sqrt{x} < \epsilon$
- Now, square both sides and we get $x < \epsilon^2$
- Does $\delta = \epsilon^2$ satisfy the definition of a limit?
- If $0 < x < \delta$, then $\sqrt{x} < \sqrt{\delta}$
- Since $\delta = \epsilon^2$, we can also say $\sqrt{x} < \sqrt{\epsilon^2}$, or $\sqrt{x} < \epsilon$
- Therefore, $\delta = \epsilon^2$ ensures $0 < x < \delta \Leftrightarrow |\sqrt{x} - 0| < \epsilon \quad \square$

From Limits to Derivatives

- Limits help us understand what goes on when we differentiate a function, or take its derivative
- Derivatives tell us the rate of change in $f(x)$ at x
- For example, suppose $f(x) = x^2$, and we wish to know how sharply the parabola is increasing from $x = 2$ to $x = 3$
- $f(x)$ increases by 5 over this 1 unit increase in x
- But we would get a different value for every 1 unit interval
- The instantaneous rate of change at any given point is the slope, m, of the tangent line at that point
- Provided it exists, $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

Defining Derivatives

- Ex: the tangent line for $f(x) = x^2$ at $x = 1$ is 2
- $lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} x + 1 = 1 + 1 = 2$
- Now suppose we wish to be able to express the instantaneous rate of change in $f(x)$ for any x
- The derivative of $f(x)$ is given by:
  - $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$
  - This can be written as $f'(x)$
  - Or $\frac{df(x)}{dx}$

Example

- Evaluate $f'(x)$ for $f(x) = x^2 - 8x + 9$
- $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
- $\Rightarrow \lim_{h \to 0} \frac{\left((x+h)^2 - 8(x+h) + 9\right) - (x^2 - 8x + 9)}{h}$
- $\Rightarrow \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 8x - 8h + 9 - x^2 + 8x - 9}{h}$
- $\Rightarrow \lim_{h \to 0} \frac{2xh + h^2 - 8h}{h} = \lim_{h \to 0} 2x + h - 8$
- $\Rightarrow 2x - 8$

Basic Derivative Rules

- Limit approach to differentiation always gives right answer
- But time consuming, so we use established rules as shortcuts
- Most important is the power rule, $\frac{dx^n}{dx} = nx^{n-1}$
- Examples: $\frac{d\sqrt{x}}{dx} = \frac{x^{-\frac{1}{2}}}{2}$, $\frac{d(-\frac{5}{8}x^{-\frac{8}{5}})}{dx} = x^{-\frac{13}{5}}$
- Constants get factored out, $\frac{dcf(x)}{dx} = c\frac{df(x)}{dx}$
- Note $\frac{dc}{dx} = 0$
- Derivative of a sum is the sum of the derivatives
- $\frac{d\left[f(x) + g(x)\right]}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$

Product rule

- Product rule: $\frac{d\left[f(x)g(x)\right]}{dx} = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx}$
- Example: $\frac{d\left[(3x^2-3)(4x^3-2x)\right]}{dx}$
- $\Rightarrow (3x^2-3)\frac{d(4x^3-2x)}{dx} + (4x^3-2x)\frac{d(3x^2-3)}{dx}$
- $\Rightarrow (3x^2-3)(12x^2-2) + (4x^3-2x)(6x)$
- $\Rightarrow (36x^4-6x^2-36x^2+6) + (24x^4-12x^2)$
- $\Rightarrow 60x^4-54x^2+6$
- To verify, calculate $f(x)\cdot g(x)$ then take derivative
- $(3x^2-3)(4x^3-2x) = 12x^5-18x^3+6x$
- $\frac{d(12x^5-18x^3+6x)}{dx} = 60x^4-54x^2+6$

Quotient rule

- Quotient rule: $\frac{d\frac{f(x)}{g(x)}}{dx} = \frac{g(x)\frac{df(x)}{dx} - f(x)\frac{dg(x)}{dx}}{g(x)^2}$, $g(x) \neq 0$
- Example:
  $\frac{d\frac{3x^2-3}{4x^3-2x}}{dx} = \frac{(4x^3-2x)\frac{d(3x^2-3)}{dx} - (3x^2-3)\frac{d(4x^3-2x)}{dx}}{(4x^3-2x)(4x^3-2x)}$
- $\Rightarrow \frac{(4x^3-2x)(6x) - (3x^2-3)(12x^2-2)}{16x^6 - 16x^4 + 4x^2}$
- $\Rightarrow \frac{(24x^4-12x^2) - (36x^4 - 42x^2 + 6)}{16x^6 - 16x^4 + 4x^2}$
- $\Rightarrow \frac{-12x^4 + 30x^2 - 6}{16x^6 - 16x^4 + 4x^2}$
- $\Rightarrow \frac{-6x^4 + 15x^2 - 3}{8x^6 - 8x^4 + 2x^2}$

Chain rule

- Chain rule: $\frac{df(g(x))}{dx} = \frac{dy}{du}\frac{du}{dx}$, where $y = f(u)$ and $u = g(x)$
- Example Let $g(x) = (4x^3 - 2x)$ and $f(u) = u^{-1}$
- Take $\frac{dg(x)^{-1}}{dx}$
  - $\Rightarrow -1(4x^3 - 2x)^{-2} \times \frac{d(4x^3-2x)}{dx}$
  - $\Rightarrow \frac{-(12x^2 - 2)}{(4x^3 - 2x)^2}$
  - $\Rightarrow \frac{-12x^2 + 2}{16x^6 - 16x^4 + 4x^2}$
  - $\Rightarrow \frac{-6x^2 + 1}{8x^6 - 8x^4 + 2x^2}$
- Since $\frac{[df(x)g(x)^{-1}]}{dx} = \frac{d\frac{f(x)}{g(x)}}{dx}$, we can use the chain rule followed by the product rule instead of using the quotient rule

Partial Derivatives

- A partial derivative, denoted ∂, measures the effect of a single variable on a function of multiple variables
- Let $y = \frac{3}{5}w^2z^5$, then $\frac{\partial y}{\partial w} = \frac{6}{5}wz^5$ and $\frac{\partial y}{\partial z} = 3w^2z^4$
- Ex: capturing quadratic relationships in regressions
- $Y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} - \beta_3 x_{2i}^2 + \epsilon$
- $\frac{\partial Y_i}{\partial x_{2i}} = \beta_2 - 2\beta_3 x_{2i}$
- When $\beta_2 < 0$ and $\beta_3 > 0$, the effect of x_{2i} is u-shaped
- When $\beta_2 > 0$ and $\beta_3 < 0$, the effect of x_{2i} is inverse-u

Maxima and Minima

- Derivatives can be used to find points of interest on a curve, like inflection points
- For $y = f(x)$, a point (x^*, y^*) is an inflection point if ∂^2s immediately on either side of the point are oppositely signed
- That is, for $x^* + \epsilon$ and $x^* - \epsilon$, $\frac{d^2y}{dx^2}$ is positive for one and negative for the other
- Further, if $f(x)$ is continuous at x^*, $\frac{d^2y}{dx^*} = 0$
- When $\frac{dy}{dx} = 0$, the function is at a maxima or minima
- We use the second derivative test to determine which $(\frac{d^2y}{dx^2} < 0 \rightarrow$ maxima, $\frac{d^2y}{dx^2} > 0 \rightarrow$ minima)

Global v Local Maxima and Minima

- Some functions have several points where $\frac{dy}{dx} = 0$
- These represent local maxima (minima)
- As we move away in either direction, the function is monotonically decreasing (increasing) up until $|x^* - \delta|$
- A global maxima represents a point such that δ is ∞
- Ex: let $y = f(x) = \frac{x}{4}^4 - 2x^3 + \frac{11x}{2}^2 - 6x + \frac{11}{4}$
  - $\frac{df(x)}{dx} = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$
  - Thus $f(x) = 0$ for $x = 1$, $x = 2$ and $x = 3$
  - $\frac{d^2 f(x)}{dx^2} = 3x^2 - 12x + 11$, so $\frac{d^2 f(x)}{dx} > 0$ for $x = 1$, $\frac{d^2 f(x)}{dx} < 0$ for $x = 2$, $\frac{d^2 f(x)}{dx} > 0$ for $x = 3$

Concavity and Convexity

- A function is concave if for any points x_1 and x_2, the line between $f(x_1)$ and $f(x_2)$ lies below the curve
- A function is convex if for any points x_1 and x_2, the line between $f(x_1)$ and $f(x_2)$ lies above the curve
- Easy to see graphically, but can also be verified algebraically
- $f(x)$ is concave over domain D iff $f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2)\ \forall\ \lambda \in [0,1]$ and all $x_1, x_2 \in D$, and convex if the left side is $\leq$ the right side
- If $f(x)$ is concave and $\dfrac{df(x^*)}{dx} = 0$, x^* is a global maximum
- If $f(x)$ is convex and $\dfrac{df(x^*)}{dx} = 0$, x^* is a global minimum

Riemann Integrals

- Can approximate the area under a curve with n rectangles
- This is called Riemman integration
  - Let each rectangle have the height equal to that of the curve
  - And let each have width $h = \dfrac{b-a}{n}$
  - Where a and b are the start and end points of the area
  - Height of first bar is a, last b, all others $f(a + ih)$
  - Must choose whether bars touch at upper-left or upper-right
  - Where $h \sum_{i=0}^{n-1} f(a + ih)$ defines the left Riemann integral
  - And $h \sum_{i=1}^{n} f(a + ih)$ defines the right

Riemann with Infinitely Small h

- The larger h (smaller n), the less accurate this procedure is
- The true area always lies in the interval bounded by the left and right Riemann integrals
- Let $S_{left} = \lim_{h \to 0} h \sum_{i=0}^{n-1} f(a + ih)$
- And let $S_{right} = \lim_{h \to 0} h \sum_{i=1}^{n} f(a + ih)$
- If $f(x)$ is continuous, we can find the area between a and b by setting $S_{left} = S_{right} = R$
- Where R, the definite integral, is equal to $\int_a^b f(x)\, dx$

Fundamental Theorem of Calculus

- Antiderivative $F(x) : \dfrac{dF(x)}{x} = f(x)$
- Mean Value Theorem: $f(b) - f(a) = \dfrac{df(\hat{x})}{dx}(b - a)$
- Then, by MVT, $F(x_i) - F(x_{i-1}) = f(\hat{x}_i)(x_i - x_{i-1})$
- Evaluate for each of H pairs of x_i, x_{i-1} from a to b:
- When we add a series of H equations together:
  - $F(x_i)\ \forall\ i \in [1, H-1]$ drops out on the left side
  - This leaves $F(b) - F(a)$ on l.h.s.
  - The r.h.s. reduces to $\sum_{i=1}^{H} f(\hat{x})(x_i - x_{i-1})$
  - As $h \to 0$, that's equivalent to $\int_a^b f(x)\ dx$
- Thus we have shown $\int_a^b f(x)\ dx = F(b) - F(a)$

Integrating with Antiderivatives

- Take $\int_1^2 (15y^4 + 8y^3 - 9y^2 + y - 3)\ dy$
- $F(y) = 3y^5 + 2y^4 - 3y^3 + \frac{y^2}{2} - 3y$ since
- $\frac{dF(y)}{dy} = \frac{d(3y^5 + 2y^4 - 3y^3 + \frac{y^2}{2} - 3y)}{dy} =$
  $15y^4 + 8y^3 - 9y^2 + y - 3$
- Therefore, $\int_1^2 f(y)\ dy = 3y^5 + 2y^4 - 3y^3 + \frac{y^2}{2} - 3y|_{y=1}^{y=2}$
- $\Rightarrow \left(3(2)^5 + 2(2)^4 - 3(2)^3 + \frac{2^2}{2} - 3(2)\right)$
  $- \left(3(1)^5 + 2(1)^4 - 3(1)^3 + \frac{1^2}{2} - 3(1)\right)$
- $\Rightarrow (96 + 32 - 24 + 2 - 6) - (3 + 2 - 3 + \frac{1}{2} - 3)$
- $\Rightarrow 100.5$

Indefinite Integrals

- When we specify the range over which we are integrating, we calculate definite integrals
- But we can also calculate indefinite integrals
- Here $\int f(x)\ dx = F(x) + C$, where C is an arbitrary constant
- Example: $\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$
- C represents the constant that would be removed by differentiating $F(x)$
- This allows us to represent functions more generally when we aren't necessarily interested in measuring specific areas
- This is particularly useful for differential equations

Integral Rules

- Constants: $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$
- Addition: $\int_a^b (f(x)+g(x))\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$
- By pieces: $\int_a^c f(x)\ dx = \int_a^b f(x)\ dx + \int_b^c f(x)\ dx$
- Limit reversal: $\int_a^b f(x)\ dx = -\int_a^b f(x)\ d(-x)$
- Integration by parts: $\int u\ dv = uv - \int v\ du$ where $u = f(x), v = g(x),\ du = \frac{df(x)}{dx},\ dv = \frac{dg(x)}{x}$

Higher Order Derivatives

- Calculate n^{th} derivative through successive differentiation
- Ex: Let $f(x) = 3x^3 + z$, $\frac{\partial f(x)}{\partial x} = 9x^2$, $\frac{\partial^2 f(x)}{\partial x^2} = 18x$, etc
- For example, if nation A has GDP growth of 9% one year, 7% the next, then 3%, it's GDP is growing at a decelerating rate, and $\frac{\partial G}{\partial t} > 0$, $\frac{\partial^2 G}{\partial t^2} < 0$, where G is GDP and t is time
- Note you can differentiate with respect to different variables, at different orders, simultaneously
- Let $f(x, y) = 3x^3y^2$,
$$\frac{\partial^3 f(x,y)}{\partial x^2 \partial y} = \frac{\partial^2(9x^2y^2)}{\partial x \partial y} = \frac{\partial(18xy^2)}{\partial y} = 36xy$$
- Exponent on ∂ in numerator indicates total number of differentiations, denominator how many for each variable

Double Integrals

- Just as integrals measure the area under a curve, double integrals measure the volume under a two-dimensional surface
- More formally, $V = \int_a^b \int_c^d f(x,y)\ dydx$
- x is integrated between a and b, y between c and d
- Is equivalent to $V = \int_a^b \left[\int_c^d f(x,y)\ dy\right]\ dx$
- Calculate one integral then the other, with second integral on new function that resulted from the first
- $V = \int_a^b g(x)\ dx$, where $g(x) = \int_c^d f(x,y)\ dy$ and x is treated as a constant when calculating inner integral

Example

$$\int_0^1 \int_2^3 x^2 y^3 \; dxdy$$

$$= \int_0^1 \left[\frac{1}{3} x^3 y^3 |_{x=2}^{x=3} \right] dy$$

$$= \int_0^1 \left[\frac{1}{3} 3^3 y^3 - \frac{1}{3} 2^3 y3 \right] dy$$

$$= \int_0^1 \frac{19}{3} y^3 \; dy = \frac{19}{12} y^4 |_{y=0}^{y=1} = \frac{19}{12}$$

Unconstrained v Constrained Optimization

- Optimization identifies maxima
- We maximize $f(x)$ by finding x^* such that $\frac{df(x)}{dx} = 0$
- Check $\frac{d^2f(x)}{dx^2}$ to be sure that you have found a maximum
- Sometimes we wish to maximize $f(x_1, x_2)$ subject to some constraints on x_1, x_2
- Usually expressed as inequalities such as $x_1 > x_2$ or specific equations $x_1 + x_2 \leq 10$
- The procedure we use here is constrained optimization
- Constrained solution is never better than the unconstrained

Constrained Optimization

- We seek to maximize $f(x)$ subject to the arbitrary constraints expressed as m functions
  - $x_1 \leq r_1, x_2 \leq r_2, \ldots, x_m \leq r_m$
  - Where $r_1, r_2, \ldots, r_m$ are constants
- For this, we use the Lagrange multiplier
  - We replace $f(x)$ with $L(x, \lambda)$
  - where $L(x, \lambda) = f(x) + \lambda_1 (x_1 - r_1) + \ldots \lambda_m (x_m - r_m)$
- To identify optimal x, x^*, and constraint coefficient:
  - First, we differentiate L w.r.t. x
  - The value of x that satisfies $\frac{\partial L}{\partial x} = 0$ is x^*
  - Provided, of course, that it is a maxima
  - Next, we differentiate L w.r.t. λ
  - Value of λ that satisfies $\frac{\partial L}{\partial \lambda}$ is the constraint coefficient
  - Iff this value is 0, constraints were not binding

Voluntary Provision of Public Goods

- Let G be quantity of public good provided by N individuals and G_i individual contributions, $\ni G = \sum_{i=1}^{N} G_i$
- Let $u_i(X_i, G)$ be utility function for i, given G and X_i consumption of private goods
- Each i is subject to budget constraint $Y_i = P_x X_i + P_g G_i$
- If public good provision is voluntary, i takes decision of all $j \in N$ into account but treats them as fixed
- Each i sets G_i such that is maximizes u_i, given G_j
- This calls for constrained optimization
- Where we rewrite the constraint as $Y_i - P_x X_i - P_g G_i = 0$
- And thus $L = u_i(X_i, G) + \lambda_i(Y_i - P_x X_i - P_g G_i)$

Optimal Voluntary Contributions

- The optimal contribution to public goods is found by taking $\frac{\partial L}{\partial G_i}$ and setting it equal to 0
  - This gives us $\frac{\partial u_i}{\partial G_i} - \lambda_i P_g = 0$
  - It will prove helpful to compare this to the optimal x_i
  - That is found in the same manner
  - We get $\frac{\partial u_i}{\partial X_i} - \lambda_i P_x = 0$
- That is, the rate at which i derives utility from G_i relative to the rate at which i derives utility from X_i, known as the marginal rate of substitution, is $\frac{P_g}{P_x}$
- Individuals treat public goods as if they are private goods, ignoring positive externalities

Pareto Optimal Contributions

- The Pareto optimal level of provision would maximize the societal welfare function
- Let this be given by $\omega = \sum_{i=1}^{N} \gamma_i U_i$
  - We restrict $\gamma_i > 0$
  - This ensures that ω cannot increase if any U_i decreases
  - Anything else would not be Pareto optimal
- Now maximize ω subject to $\sum_{i=1}^{N} Y_i = P_x \sum_{i=1}^{N} X_i + P_g G$
  - $L_\omega = \omega + \lambda(\sum_{i=1}^{N} Y_i - P_x \sum_{i=1}^{N} X_i - P_g G)$
  - For G, $\frac{\partial L_\omega}{\partial G} = \sum_{i=1}^{N} \gamma_i \frac{\partial U_i}{\partial G} - \lambda P_g = 0$
  - And, for each X_i, $\gamma_i \frac{\partial u_i}{\partial X_i} - \lambda P_x = 0$

Voluntary versus Pareto

- We can rewrite the Pareto optimal X_i's as $\gamma_i = \dfrac{\lambda P_x}{\partial u_i/\partial X_i}$
- And substituting all N of these γ_i's into the first expression
- $\displaystyle\sum_{i=1}^{N} \frac{\lambda P_x}{\partial u_i/\partial X_i} \cdot \frac{\partial u_i}{\partial G} - \lambda P_g = 0$
- $\displaystyle\Rightarrow \sum_{i=1}^{N} \frac{\lambda P_x}{\partial u_i/\partial X_i} \cdot \frac{\partial u_i}{\partial G} = \lambda P_g$
- Dividing both sides by λP_x, we obtain
- $\displaystyle\sum_{i=1}^{N} \frac{\partial u_i/\partial G}{\partial u_i/\partial X_i} = \frac{P_g}{P_x}$
- We can rewrite this as $\displaystyle\frac{\partial u_i/\partial G_i}{\partial u_i/\partial X_i} = \frac{P_g}{P_x} - \sum_{j \neq i}^{N} \frac{\partial u_j/\partial G_j}{\partial u_j/\partial X_j}$

Implications

- If G and X are normal goods, the Pareto optimal marginal rate of substitution of public for private goods is strictly less than the optimal rate under voluntary provision
- A lower marginal rate of substitution means the degree to each G increases ω relative to the degree to which X_i increases ω is lower than the degree to which G_i increases U_i relative to the degree to which X_i increases U_i
- It takes less G_i to maximize u_i than G to maximize ω
- Less G_i will be voluntarily produced than is Pareto optimal
- As N increases, the degree of underprovision increases